For Re ≫ 1 the flow rapidly goes over into the potential everywhere except the domain near the surface since as $r \to 1$ one of the factors in the exponent $\left(r - \frac{3}{4} - \frac{1}{4}\frac{1}{r^2}\right)$ tends to zero. Consequently, no matter how large the value of Re there is always a neighborhood near the surface where the exponent $\frac{\mathrm{Re}}{2}\left(r - \frac{3}{4} - \frac{1}{4}\frac{1}{r^2}\right)(\cos\theta - 1)$ will be small, and the whole viscous permutation is concentrated in this domain, as corresponds to the physical model of the flow in boundary layer theory.

On the basis of an analysis, the deduction can be made that the method considered permits, in principle, approximate solutions to be obtained for the complete Navier–Stokes equations for a sufficiently large flow range by extending the equally suitable solution obtained to the domain of larger Re by successive approximations.

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STEADY-STATE FLOW OF A RIVULET ALONG A SURFACE UNDER THE INFLUENCE OF ACCELERATION

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Consideration is given to dependence of a solution for steady-state flow of a rivulet of viscous incompressible liquid on a hard flat wall on the following independent primary parameters: density ρ (kg/m^3), kinematic viscosity ν (m^2/sec), and surface tension σ (kg/sec^2) for the liquid, contact wetting angle α at the boundary of the three media, width of the main rivulet H (m) [or flow rate in the rivulet Q (m^3/sec)], field acceleration a (m/sec^2) directed along the wall. The following assumptions were made: only velocity component v (m/sec) directed along a equals zero.

A cross section of the rivulet is a region Γ bounded by a section with length H from the direction of the wall and the arc of a circle at the free surface of the rivulet. The arc of the circle and section intersect at an angle equal to α. External pressure p_0 is constant, and tangential stresses at the free surface from the direction of the external medium are ignored [1, 2].

In region Γ we find the distribution of velocities v, in particular the maximum velocity, flow rate Q_1, momentum fluxes I, and kinetic energy G in relation to the arguments enumerated above. Balance equations for the momentum and continuity for the incompressible Newtonian liquid have the form

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p/\rho + \nu\Delta\mathbf{v} + \mathbf{a},\ \nabla\cdot\mathbf{v} = 0,\ \nu = \mu/\rho. \quad (1)$$

With the assumptions made above in a coordinate system where axis OZ is directed along **a**, (1) is brought into the form $-\nabla_z p/\rho + \nu\Delta v + a = 0$, since $(\mathbf{v}\cdot\nabla)\mathbf{v} = 0$ in view of the assumption

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that $v_x = v_y = 0$, and $\partial v/\partial z = 0$ in view of $\nabla \cdot \mathbf{v} = 0$.

The condition $v = 0$ is fulfilled at the wall, and at the free surface the rivulet fulfills the condition for balance of momentum [3]

$$[-p\delta_{ij} + \mu(\nabla_i v_j + \nabla_j v_i)]_1^2 n_j = -\sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)n_i, \tag{2}$$

where normal n_i is directed from the rivulet; $[\ ]_1^2$ indicates the difference in values in the external medium and in the rivulet. In a cylindrical coordinate system which is selected so that one of the coordinate surfaces coincides with the free surface of the rivulet, only the viscous stress tensor components differ from zero $\tau_{13} = \tau_{31} = \mu\partial v/\partial r$ and $\tau_{23} = \tau_{32} = (\mu/r)\partial v/\partial\varphi$. Projection of Eq. (2) on normal n_i gives $-p + p_0 = -\sigma/R$, since $\tau_{ij}n_j n_i = 0$ (R is radius of curvature of the free rivulet surface). Projection of Eq. (2) on axis OZ gives $\mu\partial v/\partial r = 0$.

Thus, in the region Γ, $p = p_0 + \sigma/R$ and $\nabla p = 0$, i.e., $\nu\Delta v + a = 0$. At the wall $v = 0$, and at the free surface $\partial v/\partial n = 0$.

We take as scales H and a/ν, then

$$v = (a/\nu)H^2 UH(\alpha),\ Q = (a/\nu)H^4 QH(\alpha),$$
$$I = \rho\frac{a^2}{\nu^2}H^6 IH(\alpha),\quad G = \rho\frac{a^3}{2\nu^3}H^8 GH(\alpha), \tag{3}$$

where $UH(\alpha)$, $QH(\alpha)$, $IH(\alpha)$, $GH(\alpha)$ are dimensionless velocity, flow rate, momentum flux, and kinetic energy of the rivulet with an individual base.

We increase dimensionless region Γ with an individual base by a factor of two and we place it on a complex plane $z = (x, y)$ with $\operatorname{Im} z \geqslant 0$ so that angular points of it are found at points $(-1, 0)$ and $(1, 0)$, and the center of the section is found at the origin. We introduce a new variable $u = 4UH$, then $\Delta u = -1$ in region Γ.

We present the solution of u as the sum of partial solution $-y^2/2$ and total solution v: $u = -y^2/2 + v$. Then in region Γ, $\Delta v = 0$, and $v = 0$ in section $[-1, 1]$ of the actual axis. In the arc

$$\partial v/\partial n = [-\mathrm{grad}\,(-y^2/2)]\mathbf{n} = y(y\sin\alpha + \cos\alpha), \tag{4}$$

since $\mathbf{n} = (x\sin\alpha,\ y\sin\alpha + \cos\alpha)$.

In order to find the harmonic function of v in region Γ with displacement of boundary conditions (4) we continue v analytically into region Γ' which is symmetrical to region Γ relative to section $[-1, 1]$ of the actual axis with the condition $v(x, -y) = -v(x, y)$. This may be done [4] since $v(x, y) = 0$ with $y = 0$, and derivatives are continuous in section $[-1, 1]$. In regions Γ' and Γ boundary conditions are connected by the relationship

$$\partial v(x,\ -y)/\partial n = -\partial v(x,\ y)/\partial n = y(y\sin\alpha + \cos\alpha),\ y \geqslant 0. \tag{5}$$

Finding the harmonic function in region $\Gamma + \Gamma'$ with conditions (5) is equivalent to finding the conformal representation of region $\Gamma + \Gamma'$ in any canonical region for which the solution method is known if normal derivatives are prescribed for the harmonic function at the boundary of the canonic region [4].

We shall find the conformal representation of an individual circle of complex region w in region $\Gamma + \Gamma'$ in complex plane z. Transform $z_1 = (1 + w)/(1 - w)$ converts the region within the individual circle from plane w into a right hemiplane in z_1. Transform $z_2 = z_1^c$, where $c = 2\alpha/\pi$, converts the hemiplane from z_1 into a region bounded by rays emerging from the origin with angles $-\alpha$ and α in plane z_2. Transform $z_3 = (z_2 - 1)/(z_2 + 1)$ converts the angle formed by the rays from plane z_2 into region $\Gamma + \Gamma'$ in plane z. Finally transform

$$z(w) = z_3\{z_2[z_1(w)]\} = \frac{(1+w)^c - (1-w)^c}{(1+w)^c + (1-w)^c} \tag{6}$$

converts the individual circle from plane w into region $\Gamma + \Gamma'$ in plane z. Points of the real axis -1, 0, 1 remain in place. The section of the imaginary axis from 0 to i is converted into the height of the rivulet. Transform (6), with the exception of angular points, is conformal, i.e., $\Delta v = 0$ also in circle $|w| < 1$.

We introduce at the boundary of an individual circle parameter $w = e^{i\tau}$. Transform (6) determines at the boundary of the region $\Gamma + \Gamma'$ the dependence of z, y, and $\partial v/\partial n$ on parameter τ:

TABLE 1

α, deg	QH	IH	GH	UHM
30	$3{,}2\cdot10^{-4}$	$1{,}9\cdot10^{-6}$	$1{,}3\cdot10^{-8}$	$8{,}1\cdot10^{-3}$
60	$3{,}9\cdot10^{-3}$	$9{,}7\cdot10^{-5}$	$2{,}6\cdot10^{-6}$	$3{,}4\cdot10^{-2}$
90	$2{,}4\cdot10^{-2}$	$1{,}8\cdot10^{-3}$	$1{,}5\cdot10^{-4}$	$9{,}4\cdot10^{-2}$
120	0,19	$4{,}8\cdot10^{-2}$	$1{,}3\cdot10^{-2}$	0,29
150	5,3	8,6	14,4	1,55

$$z(\tau)=\left[e^{i\alpha}-\left(\operatorname{tg}\frac{\tau}{2}\right)^{c}\right]\Big/\left[e^{i\alpha}+\left(\operatorname{tg}\frac{\tau}{2}\right)^{c}\right]; \tag{7}$$

$$y(\tau)=2\sin\alpha\left[\left(\operatorname{ctg}\frac{\tau}{2}\right)^{c}+2+\left(\operatorname{tg}\frac{\tau}{2}\right)^{c}\right]^{-1},\quad \partial v/\partial n=y\,(y\sin\alpha+\cos\alpha). \tag{8}$$

It is well known [4] that with conformal transform $\partial v(w)/\partial n = \partial v(z)/\partial n|\partial z/\partial w|$, where $w = e^{i\tau}$, and z is determined by (7):

$$\left|\frac{\partial z}{\partial w}\right|=\frac{4c\,(1+w)^{c-1}(1-w)^{c-1}}{[(1+w)^{c}+(1-w)^{c}]^{2}}=c\left\{\left[\left(\operatorname{tg}\frac{\tau}{2}\right)^{c}+2\cos\alpha+\left(\operatorname{ctg}\frac{\tau}{2}\right)^{c}\right]\sin\frac{\tau}{2}\cos\frac{\tau}{2}\right\}^{-1}.$$

Solution of the problem v(w) in an individual circle, when the normal derivative $\partial v/\partial n(e^{i\tau})$ at the boundary is prescribed, is found by the Din equation [4]

$$v(w)=-\frac{1}{\pi}\int_{0}^{2\pi}\frac{\partial v}{\partial n}(e^{i\tau})\ln|e^{i\tau}-w|\,d\tau$$

($|e^{i\tau} - w|$ is the distance from point w to a variable point of the boundary at the circle).

The general solution for u in the upper half of the individual circle in plane w takes the form

$$u(w)=-y^{2}/2-\frac{1}{\pi}\int_{0}^{2\pi}\left|\frac{\partial z}{\partial w}\right|\frac{\partial v(z)}{\partial n}\ln|e^{i\tau}-w|\,d\tau.$$

Flow rate QH(α), momentum flux IH(α), and kinetic energy GH(α) through the rivulet cross section are determined by the expressions

$$\int_{0}^{1}\int_{0}^{\pi}u^{k}\left|\frac{\partial z}{\partial w}\right|^{2}r\,dr\,d\varphi,\quad k=1,2,3 \tag{9}$$

($\left|\frac{\partial z}{\partial w}\right|^{2}r\,dr\,d\varphi$ is area of an element in plane z).

A program was composed and calculation was carried out in an ES-1040 computer form QH(α), IH(α), GH(α), and UHM in relation to α (Table 1).

In solving the stated problem in dimensionless form a, ν, and H were selected as scales. However, instead of H it is possible to specify Q, and H becomes unknown. Then from (3) we obtain $Q=\frac{a}{\nu}H^{4}QH(\alpha)$ or $H=\left(\frac{Q\nu}{a}\right)^{1/4}[QH(\alpha)]^{-1/4}=\left(\frac{Q\nu}{a}\right)^{1/4}HQ(\alpha)$, where $(Q\nu/a)^{1/4}$ is scale of length; $HQ(\alpha)=[QH(\alpha)]^{-1/4}$ is dimensionless width of the main rivulet. Similarly from (3) and (9) we have

$$U=\left(\frac{Qa}{\nu}\right)^{1/2}UQ(\alpha),\quad I=\rho\left(\frac{Q^{3}a}{\nu}\right)^{1/2}IQ(\alpha),\quad G=\rho\frac{Q^{2}a}{2\nu}GQ(\alpha).$$

Here $\left(\frac{Qa}{\nu}\right)^{1/2}$, $\rho\left(\frac{Q^{3}a}{\nu}\right)^{1/2}$, $\rho\frac{Q^{2}a}{2\nu}$ are scales for velocity, momentum flux, and kinetic energy; UQ(α), IQ(α), GQ(α) are their corresponding dimensionless parameters (Table 2).

In Table 1 parameters with varying contact wetting angles from 30 to 150° change by a few orders of magnitude, and in Table 2 they vary within the limits of an order of magnitude, differing from unity by not more than an order of magnitude. This means it is possible to assume that $(Qa/\nu)^{1/4}$, $\rho(Q^{3}a/\nu)^{1/2}$, $\rho Q^{2}a/2\nu$ determine the scales for velocity, momentum flux, and kinetic energy in a rivulet over a wide range of change in α.

TABLE 2

α, deg	HQ	IQ	GQ	UHM
30	7,4	0,33	0,12	0,45
60	4,0	0,39	0,17	0,53
90	2,5	0,47	0,25	0,60
120	1,5	0,57	0,35	0,66
150	0,66	0,70	0,51	0,67

TABLE 3

α, deg	$H\cdot 10^3$, m	U, m/sec	$H\cdot 10^5$, m	U, m/sec
	Water		Nickel	
30	4,2	1,25	21	0,59
60	2,3	1,7	11	0,69
90	1,4	1,9	6,9	0,79
120	0,84	2,1	4,1	0,86
150	0,37	2,1	1,8	0,86

As an example we calculate the flow parameters for rivulets of water over a vertical wall and rivulets of molten nickel on a rapidly rotating horizontal disk. For water $\rho = 10^3$ kg/m^3, $\nu = 10^{-6}$ m^2/sec, $a = 9.8$ m/sec^2, $Q = 10^{-6}$ m^3/sec, and for nickel at 1800 K, $\nu = 6.4\cdot 10^{-7}$ m^2/sec, disk radius is 10^{-1} m, and the number of revolutions is 104.7 rad/sec, $Q = 10^{-9}$ m^3/sec (Table 3). The radio of Coriolis to centrifugal acceleration is estimated from

$$\frac{2\omega v}{\omega^2 r} = \frac{2v}{\omega r} = 2\left(\frac{Qa}{\nu}\right)^{1/2}\frac{UQ(\alpha)}{\omega r} = 2\left(\frac{Q}{\nu r}\right)^{1/2} UQ(\alpha), \quad a = \omega^2 r.$$

The maximum of ratio $2v/\omega r = 0.16$ is achieved with $\alpha = 150°$. In [1, 2, 5, 6] the two-dimensional velocity field is found in the form $v = v[y/\delta(x)]$, where the parabolic profile is taken from the solution for the unidimensional problem for a film. With this choice of velocity field the boundary condition at the free rivulet surface is not observed.

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PERTURBATION METHOD COMPUTATION OF THE MAXIMAL GROUP VELOCITIES OF INTERNAL WAVES IN A STRATIFIED MEDIUM WITH MEAN SHEAR FLOWS

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Propagation of internal gravitational waves excited in a stratified fluid layer $-H < z < 0$, $-\infty < x, y < \infty$ with mean horizontal shear flows is described by the equation [1]

$$Lu(t, x, y, z, z_0) = Q(t, x, y, z, z_0), \; u = 0(z = 0, -H), \tag{1}$$

where the operator is

$$L = \frac{D^2}{Dt^2}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] - \frac{D}{Dt}\left[U''_{zz}\frac{\partial}{\partial x} + V''_{zz}\frac{\partial}{\partial y}\right] + N^2\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right];$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y};$$

$U = U(z)$, $V = V(z)$ are the velocity components of the mean flow $\mathbf{U} = \{U, V, 0\}$ at the horizon z, and $N(z)$ is the Brunt–Väisälä frequency. The Boussinesq and solid covers are used. The Miles stability condition $\mathrm{Ri}(z) = N^2(z)/[(U'_z)^2 + (V'_z)^2] > 1/4$ is assumed satisfied and

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